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CENTER OF FLEXURE OF A HOLLOW COMPOUND CANTILEVER

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It was noted in [1] that the center of flexure of a solid cantilever can be determined if the solution of the problem of torsion of this cantilever is available. This assertion was generalized in [2] to the case of a hollow cantilever. In the present paper the formulas obtained earlier [3,4] for the coordinates of the center of flexure of a hollow cantilever are generalized with the help of a complex torsional function, to the case of a hollow compound cantilever.

1. Let us consider a cantilever composed of multiconnected prismatic bodies made of different materials and joined together along their lateral surfaces. Then the region occupied by any transverse cross section of the cantilever will consist of piecewise different inclusions S_j $(j = 0,1, \ldots, m)$ where m is the number of inclusions within the region S_0 with the lines of contact $L_{0q}{}^*$ $(q = 1, 2, \ldots, m)$. We also

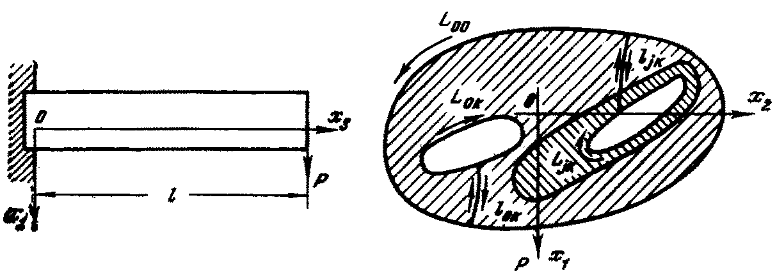

Fig. 1

denote the contours of the cutouts of the inclusions S_j by L_{jk} $(k = 0, 1, 2, \ldots, N_j)$ where N_j is the number of the inclusion cutouts. We introduce the following notation:

$$L = \sum_{j=0}^{m} \sum_{k=0}^{N_j} L_{jk}, \quad l = \sum_{j=0}^{m} \sum_{q=1}^{N_j} l_{jk}$$

$$L' = L + l, \quad \omega = \sum_{j=0}^{m} \omega_j$$

Here L denote the sum of the contours of the regions S_j, l_{jk} are the cut lines, L'

is the contour of a singly connected region and ω_j is the area of the region.

Let a prismatic body of length l be clamped at one end and subjected at the other end to a load statically equivalent to a force P. We place the coordinate origin at any point of the clamped end. The Ox_3-axis is directed parallel to the cantilever axis and the Ox_1-axis is parallel to the force P (see Fig. 1). The components of the stress tensor generated by the bending of the cantilever are sought in the form [1]

$$\begin{aligned} &\sigma_{11} = \sigma_{22} = \sigma_{12} = 0 \\ &\sigma_{33} = P\,(ax_1 + bx_2 + e)\,(l - x_3) \\ &\sigma_{31} = \frac{P}{2}\left(\frac{\partial \chi_j}{\partial x_2} + ax_1^2 + ex_1\right) \\ &\sigma_{32} = \frac{P}{2}\left(-\frac{\partial \chi_j}{\partial x_1} + bx_2^2 + ex_2\right) \end{aligned} \tag{1.1}$$

Here $\chi = \chi\,(x_1, x_2)$ is an unknown function, and a, b and e are unknown constants.

The components of the stress tensor σ_{33}, σ_{31} and σ_{32} must satisfy the following conditions of equilibrium in the x_3 cross section:

$$\sum_{j=0}^{m}\int_{\omega_j} \sigma_{31} d\omega - P = 0, \quad \sum_{j=0}^{m}\int_{\omega_j} \sigma_{32} d\omega = 0 \tag{1.2}$$

$$\begin{aligned} &\sum_{j=0}^{m}\int_{\omega_j} (x_1\sigma_{32} - x_2\sigma_{31})\, d\omega = 0 \\ &\sum_{j=0}^{m}\int_{\omega_j} \sigma_{33} d\omega = 0, \quad \sum_{j=0}^{m}\int_{\omega_j} \sigma_{33} x_2 d\omega = 0 \\ &\sum_{j=0}^{m}\int_{\omega_j} \sigma_{33} x_1 d\omega + P\,(l - x_3) = 0 \end{aligned} \tag{1.3}$$

Substituting the second relation of (1.1) into the conditions (1.3), we obtain a linear algebraic system the roots of which are

$$a = -\frac{I_{11}\omega - S_1^2}{B}, \quad b = \frac{I_{12}\omega - S_1 S_2}{B}, \quad e = \frac{I_{11}S_2 - I_{12}S_1}{B}$$

$$I_{\alpha\beta} = \sum_{j=0}^{m} I_{j,\,\alpha\beta}, \quad S_\alpha = \sum_{j=0}^{m} S_{j,\,\alpha}; \quad \alpha, \beta = 1, 2$$

$$B = \begin{vmatrix} I_{22} & I_{12} & S_2 \\ I_{12} & I_{11} & S_1 \\ S_2 & S_1 & \omega \end{vmatrix}$$

Here $I_{j,\,\alpha\beta}$ and $S_{j,\,\alpha}$ are the moments of inertia and the static moments of the area ω_j relative to the x_1- and x_2- axes.

The last two relations of (1.1) satisfy the differential equations of equilibrium identically. When the relations (1.1) hold, the six Beltrami-Mitchell expressions yield the following expression for the function χ_j (here C is the constant of integration):

$$\Delta\chi_j = \frac{2\nu_j}{1+\nu_j}(bx_1 - ax_2) - 2C \qquad (1.4)$$

The conditions of zero loading at the lateral surface of the cantilever yield a boundary condition of the form

$$\frac{\partial\chi_j}{\partial l} = (bx_2^2 + ex_2)\frac{dx_1}{dl} - (ax_1^2 + ex_1)\frac{dx_2}{dl} \text{ на } L \qquad (1.5)$$

2. Let us write the function χ_j in the form

$$\chi_j = \Psi_j + C\Phi_j \qquad (2.1)$$

Then the following problems arise for the functions Φ_j and Ψ_j :

$$\Delta\Phi_j = -2 \text{ in } S_j \qquad (2.2)$$

$$\frac{\partial\Phi_j}{\partial l} = 0 \quad \text{на} \quad L_{jk}$$

$$\Delta\Psi_j = \frac{2\nu_j}{1+\nu_j}(bx_1 - ax_2) \text{ в } S_j \qquad (2.3)$$

$$\frac{\partial\Psi_j}{\partial l} = (bx_2^2 + ex_2)\frac{dx_1}{dl} - (ax_1^2 + ex_1)\frac{dx_2}{dl} \text{ on } L_{jk}$$

where Φ_j is the Prandtl stress function and Ψ_j is the flexure function. The first and second condition of (1.2) are satisfied identically.

To find the torsional moment M_{ck} we substitute the two last relations of (1.1) into the left-hand side of the condition (1.2) and take into account (2.1). After a series of manipulations we obtain

$$M_{ck} = P\Big\{C\sum_{j=0}^{m}\int_{\omega_j}\Phi_j d\omega + \sum_{j=0}^{m}\int_{\omega_j}\Psi_j d\omega + \qquad (2.4)$$

$$\frac{1}{2}\sum_{j=0}^{m}\int_{\omega_j}(bx_2 - ax_1)x_1x_2 d\omega + \int_{L'}\omega_{jk}{}^l\frac{\partial\Psi_j}{\partial l}dl -$$

$$[\Psi_j\omega_{jk}{}^l]_{L'} - C\int_{L'}\Phi_j\frac{d\omega_{jk}{}^l}{dl}dl\Big\}$$

$$\omega_{jk}{}^l = \frac{1}{2}\int_0^{^lL_{jk}}(x_1dx_2 - x_2dx_1)$$

where the symbol $[\ldots]_{L'}$ denotes the increment in the value of the function within the bracket during a single passage around the contour L'.

3. Let us recall certain relations which shall be used later.

The mean value of the torsion τ for the whole transverse cross section is given, according to [1], by the formula

$$\tau = \sum_{j=0}^{m} \frac{1}{\omega_j} \int_{\omega_j} \frac{\partial \omega_3}{\partial x_3} d\omega = \sum_{j=0}^{m} \frac{1}{\omega_j} \int_{\omega_j} \left(\frac{\partial e_{23}}{\partial x_1} - \frac{\partial e_{31}}{\partial x_2} \right) d\omega$$

Using Hooke's law and the last two formulas of (1. 1) and taking (1. 4) into account, we obtain

$$\tau = P(aS_1 - bS_2 + C\varkappa_0) \tag{3.1}$$

$$S_1 = \sum_{j=0}^{m} \frac{\nu_j}{E_j} x_{2c}^{\,j}, \quad S_2 = \sum_{j=0}^{m} \frac{\nu_j}{E_j} x_{1c}^{\,j}, \quad \varkappa_0 = \sum_{j=0}^{m} \frac{1+\nu_j}{E_j}$$

where $(x_{1c}^{\,j}, x_{2c}^{\,j})$ are the coordinates of the center of gravity of the area ω_j relative to the x_1- and x_2 axes.

We see from the formula (3. 1) that the cantilever will experience pure flexure without torsion if the constant C is found from the formula

$$C = \frac{1}{\varkappa_0}(bS_2 - aS_1) \tag{3.2}$$

Using Green's formula

$$\int_{\omega_*} (U_j \Delta V_j - V_j \Delta U_j)\, d\omega = \int_{L_*} \left(U_j \frac{\partial v_j}{\partial n} - V_j \frac{\partial u_j}{\partial n} \right) dl \tag{3.3}$$

for $U_j = 1$ and $V_j = \Psi_j$ and taking into account the first formula of (2. 3), we obtain

$$\int_{L_*} \frac{\partial \Psi_j}{\partial n} dl = - \frac{2\nu}{1+\nu} (ax_{2c} - bx_{1c})\, \omega_* \tag{3.4}$$

Here L_* denotes an arbitrary closed contour traversed anticlockwise, lying in the cross section of the cantilever, and x_{1c}, x_{2c} are the coordinates of the center of gravity of the area ω_* contained within L_*.

The above formula must hold for each internal contour within the cross section, and also for the outer contour of the cross section.

Using the fact that the function Φ_j is single-valued and, that $\omega_{jk}^{\,l-} = \omega_{jk}^{\,l+} - \omega_{jk}$, $dl_{jk}^{+} = -dl_{jk}^{-}$, we find that

$$\int_{l} \Phi_j \frac{\partial \omega_{jk}}{\partial l} dl = 0 \tag{3.5}$$

In addition to the formulas (3.2), (3.4) and (3.5) we have

$$\int_{l_{jk}} \Phi_j \frac{\partial \Psi_j}{\partial n} dl = 0, \quad \int_{l_{jk}} \varphi_j \frac{\partial \Psi_j}{\partial l} dl = 0 \tag{3.6}$$

the validity of which shall be shown below.

Let us introduce a new harmonic function

$$\Psi_{j1} = \Psi_j - \frac{\nu_j}{1+\nu_j}(bx_2{}^2 x_1 - ax_1{}^2 x_2) \tag{3.7}$$

We also have [5]

$$\frac{\partial \Phi_j}{\partial x_1} = -\frac{\partial \varphi_j}{\partial x_2} - x_1, \quad \frac{\partial \Phi_j}{\partial x_2} = \frac{\partial \varphi_j}{\partial x_1} - x_2 \tag{3.8}$$

The projection of the displacement on the Ox_3 axis can be found from the formula [5]

$$u_{3j} = u_{3j}{}^\circ + \overset{\circ}{\omega}_{knj}(x_n{}' - x_n{}^\circ) + \int_{M_0}^{M'} \left[e_{3m} + (x_n{}' - x_n)\left(\frac{\partial e_{3m}}{\partial x_n} - \frac{\partial e_{nm}}{\partial x_3}\right)\right] dx_m \tag{3.9}$$

We can assume, without affecting the generality, that $u_{3j}{}^\circ = \omega_{knj}{}^\circ = 0$.

Taking into account in (3.9) Hooke's law, the two last formulas (1.1), (2.1) and (3.8) and introducing the harmonic function Ψ_{j2}, conjugate to Ψ_{j1}, we finally obtain, after integration

$$u_{3j} = \frac{\nu_j + 1}{E_j} P(-\Psi_{j2} + C\varphi_j) + \frac{\nu_j + 1}{2E_j}\Big[(x_1{}' - x_1)\frac{\partial}{\partial x_1}(-\Psi_{j2} + c\varphi_j) + (x_2{}' - x_2)\frac{\partial}{\partial x_2}(-\Psi_{j2} + C\varphi_j) + F(x_1, x_2, x_3)$$

where $(F(x_1, x_2, x_3)$ is a polynomial.

Since U_{3j} and φ_j are singlevalued functions, so is Ψ_{j2}. Thus $\partial\Psi_{j1} / \partial l$ and $\partial\Psi_{j1} / \partial n$ and consequently $\partial\Psi_j / \partial l$ and $\partial\Psi_j / \partial n$ are equal to each other along the cut edges, which completes the proof.

4. Applying the Green's formula (3.3) to the functions Φ_j and Ψ_j and taking into account the first equations fo (2.2) and (2.3), we obtain

$$\sum_{j=0}^{m} \frac{2\nu_j}{1+\nu_j} \int_{\omega_j} (bx_1 - ax_2)\Phi_j d\omega + 2\sum_{j=0}^{m}\int_{\omega_j} \Psi_j d\omega = \int_{L'} \left(\Phi_j \frac{\partial \Psi_j}{\partial n} - \Psi_j \frac{\partial \Phi_j}{\partial n}\right) dl$$

Taking due account in the above relations of (3.4)–(3.6), (3.8) and of the fact that on traversing the contour L' the increments $[\varphi_j \Psi_j]_{L'} = 0$ and $\int_l \varphi_j (\partial\Psi_j / \partial l)\, dl = 0$, we obtain

$$\sum_{j=0}^{m} \frac{2\nu_j}{1+\nu_j} \int_{\omega_j} (bx_1 - ax_2)\, \Phi_j d\omega + 2 \sum_{j=0}^{m} \int_{\omega_j} \Psi_j d\omega =$$

$$\frac{2\nu_0}{1+\nu_0} \Big[- C_{00}\omega_{00} (ax_{2c}{}^{00} - bx_{1c}{}^{00}) + \sum_{k=1}^{N_0} C_{0k} (ax_{2c}{}^{0k} -$$

$$bx_{1c}{}^{0k})\, \omega_{0k} \Big] + \sum_{j=1}^{m} \frac{2\nu_j}{1+\nu_j} \sum_{k=0}^{N_j} C_{jk} (ax_{2c}{}^{jk} - bx_{1c}{}^{jk})\, \omega_{jk} +$$

$$2\, [\Psi_j \omega_{jk}{}^l]_{L'} - \int_L \varphi_j \frac{\partial \Psi_j}{\partial l}\, dl - 2 \int_{L'} \omega_{jk}{}^l \frac{\partial \Psi_j}{\partial l}\, dl$$

where C_{jk} denote the values of Φ_j on the contours L_{jk}, ω_{jk} is the area enclosed within the contour L_{jk} and $(x_{1c}{}^{jk},\ x_{2c}{}^{jk})$ are the coordinates of the center of gravity of the area ω_{jk}. This gives the second term of the last equation which, on substitution into (2.4), yields

$$M_{ck} = P \Big\{ C \sum_{j=0}^{m} \int_{\omega_j} \Phi_j d\omega + \frac{1}{2} \sum_{j=0}^{m} \int_{\omega_j} (bx_2 - ax_1)\, x_1 x_2 d\omega - \tag{4.1}$$

$$\sum_{j=0}^{m} \frac{\nu_j}{1+\nu_j} \int_{\omega_j} (bx_1 - ax_2)\, \Phi_j d\omega + \frac{\nu_0}{1+\nu_0} \Big[- (ax_{2c}{}^{00} - bx_{1c}{}^{00}) \times$$

$$C_{00}\omega_{00} + \sum_{k=1}^{N_0} (ax_{2c}{}^{0k} - bx_{1c}{}^{0k})\, C_{0k}\omega_{0k} \Big] +$$

$$\sum_{j=1}^{m} \frac{\nu_j}{1+\nu_j} \sum_{k=1}^{N_j} (ax_{2c}{}^{jk} - bx_{1c}{}^{jk})\, C_{jk}\omega_{jk} - \frac{1}{2} \int_L \varphi_j \frac{\partial \Psi_j}{\partial l}\, dl -$$

$$C \int_{L'} \Phi_j \frac{d\omega^l{}_{jk}}{dl}\, dl$$

Using the boundary condition (2.3) and the Gauss-Ostrogradskii formula and taking (3.8) into account, we find that in the last formula

$$\int_L \varphi_j \frac{\partial \Psi_j}{\partial l}\, dl = -2 \sum_{j=0}^{m} \int_{\omega_j} (ax_1 + bx_2 + e)\, \varphi_j d\omega - \tag{4.2}$$

$$\sum_{j=0}^{m} \int_{\omega_j} \Big[(ax_1^2 + ex_1) \frac{\partial \varphi_j}{\partial x_1} + (bx_2^2 + ex_2) \frac{\partial \varphi_j}{\partial x_2} \Big] d\omega =$$

$$-2 \sum_{j=0}^{m} (ax_1 + bx_2 + e)\, \varphi_j d\omega - \sum_{j=0}^{m} \int_{\omega_j} (ax_1 - bx_2)\, x_1 x_2 d\omega$$

while in (4.1) the relation (3.5) yields

$$\int_{L'} \Phi_j \frac{d\omega_{jk}^l}{dl} dl = \int_{L} \Phi_j \frac{d\omega_{jk}^l}{dl} dl + \int_{l} \Phi_j \frac{d\omega_{jk}^l}{dl} dl = C_{00}\omega_{00} - \sum_{j=0}^{m} \sum_{k=1}^{N_j} C_{jk}\omega_{jk} \tag{4.3}$$

The cantilever will experience pure bending without torsion, if the constant C is found from the formula (3.2).

Substituting (3.2), (4.2) and (4.3) into (4.1) we obtain the torsional moment M_{ck} which, together with the force P acting along the Ox_1-axis at the point O will cause torsionless bending of the cantilever. The moment is equal to

$$M_{ck} = P\Big\{\sum_{j=0}^{m}\Big[\frac{1}{\varkappa_0}(bS_2 - aS_1)\int_{\omega_j} \Phi_j d\omega - \frac{\nu_j}{1+\nu_j}\int_{\omega_j}(bx_1 - ax_2)\Phi_j d\omega + \int_{\omega_j}(ax_1 + bx_2 + e)\varphi_j d\omega\Big] - \Big[\frac{\nu_0}{1+\nu_0}(ax_{2c}^{\infty} - bx_{1c}^{oo}) + \frac{1}{\varkappa_0}(bS_2 - aS_1)\Big]c_{00}\omega_{00} + \sum_{k=1}^{N_0}\Big[\frac{\nu_0}{1+\nu_0}(ax_{2c}^{ok} - bx_{1c}^{ok}) + \frac{1}{\varkappa_0}(bS_2 - aS_1)\Big]C_{0k}\omega_{0k} + \sum_{j=1}^{m}\sum_{k=1}^{N_j}\Big[\frac{\nu_j}{1+\nu_j}(ax_{2c}^{jk} - bx_{1c}^{jk}) + \frac{1}{\varkappa_0}(bS_2 - aS_1)\Big]C_{jk}\omega_{jk}\Big\} \tag{4.4}$$

Let us add the force P to the torsional moment M_{ck} as given by (4.4). Then the coordinate of the point of application of the resultant force $x_{2o} = -M_{ck} / P$. Using the expression for the complex torsional potential $F(z) = -i(\varphi + i\psi)$ where $\psi = \Phi + {}^1/_2 (x_1^2 + x_2^2)$ and taking into account (4.4), we obtain the following final expression for the coordinate x_{2o} :

$$x_{2o} = \sum_{j=0}^{m}\int_{\omega_j}(ax_1 + bx_2 + e)\,\mathrm{Im}\,F_j(z)\,d\omega - \sum_{j=0}^{m}\Big\{\frac{1}{\varkappa_0}(bS_2 - aS_1)\int_{\omega_j}\Big[\mathrm{Re}\,F_j(z) - \frac{1}{2}(x_1^2 + x_2^2)\Big]d\omega - \frac{\nu_j}{1+\nu_j}\int(bx_1 - ax_2)\Big[\mathrm{Re}\,F_j(z) - \frac{1}{2}(x_1^2 + x_2^2)\Big]d\omega\Big\} + \sum_{j=0}^{m}\sum_{k=0}^{N_j}\delta_{jk}\Big[\frac{\nu_j}{1+\nu_j}(ax_{2c}^{jk} - bx_{1c}^{jk}) + \frac{1}{\varkappa_0}(bS_2 - aS_1)\Big]C_{jk}\omega_{jk} \tag{4.5}$$

$$\delta_{jk} = \begin{cases} 1, & \text{when } j = k = 0 \\ -1, & \text{otherwise} \end{cases}$$

Let us now direct the force P parallel to the Ox_2 axis. This will give, in the same manner, the coordinate $x_{1o} = M_{ck}^*/P$ or finally,

$$x_{1o} = -\sum_{j=0}^{m}\int_{\omega_j}(a_* x_1 + b_* x_2 + e_*)\,\mathrm{Im}\,F_j(z)\,d\omega + \sum_{j=0}^{m}\Big\{\frac{1}{\varkappa_0}(b_* S_2 - a_* S_1)\int_{\omega_j}\Big[\mathrm{Re}\,F_j(z) - \frac{1}{2}(x_1^2 + x_2^2)\Big]d\omega - \frac{\nu_j}{1+\nu_j}\int_{\omega_j}(b_* x_1 - a_* x_2)\Big[\mathrm{Re}\,F_j(z) - \frac{1}{2}(x_1^2 + x_2^2)\Big]d\omega\Big\} - \sum_{j=0}^{m}\sum_{k=0}^{N_j}\delta_{jk}\Big[\frac{\nu_j}{1+\nu_j}(a_* x_{2c}^{jk} - b_* x_{1c}^{jk}) + \frac{1}{\varkappa_0}(b_* S_2 - a_* S_1)\Big]C_{jk}\omega_{jk} \tag{4.6}$$

$$a_* = \frac{I_{12}\omega - S_1 S_2}{B}, \quad b_* = \frac{S_2^2 - \omega I_{22}}{B}, \quad e_* = \frac{I_{22}S_1 - I_{12}S_2}{B}$$

The coordinates (x_{1o}, x_{2o}) are called coordinates of the center of flexure.

In the case when the cantilever is made of a homogeneous material, the formulas for the coordinates of the center of flexure (4.5) and (4.6) will become, using notation $x_{1c}^{jk} = x_{1c}^{k}$, $x_{2c}^{jk} = x_{2c}^{k}$, $C_{jk} = C_k$, $\omega_{jk} = \omega_k$

$$x_{2o} = \int_{\omega}(ax_1 + bx_2 + e)\,\mathrm{Im}\,F(z)\,d\omega + \frac{\nu_0}{1+\nu_0}\Big\{\int_{\omega}\Big[b(x_1 - x_{1c}) - a(x_2 - x_{2c})\Big(\mathrm{Re}\,F(z) - \frac{1}{2}(x_1^2 + x_2^2)\Big)\Big]d\omega - \Big[b(x_{1c}^{\circ} - x_{1c}) - a(x_{2c}^{\circ} - x_{2c})C_0\omega_0\Big] + \sum_{k=1}^{n}[b(x_{1c}^{k} - x_{1c}) - a(x_{2c}^{k} - x_{2c})]C_k\omega_k\Big\}$$

$$x_{1o} = -\int_{\omega}(a_* x_1 + b_* x_2 + e_*)\,\mathrm{Im}\,F(z)\,d\omega - \frac{\nu_0}{1+\nu_0}\Big\{\int_{\omega}b_*(x_1 - x_{1c}) - a_*(x_2 - x_{2c})\Big[\mathrm{Re}\,F(z) - \frac{1}{2}(x_1^2 + x_2^2)\Big]d\omega - [b_*(x_{1c}^{\circ} - x_{1c}) - a_*(x_{2c}^{\circ} - x_{2c})]C_0\omega_0 + \sum_{k=1}^{n}[b_*(x_{1c}^{k} - x_{1c}) - a_*(x_{2c}^{k} - x_{2c})]C_k\omega_k\Big\}$$

The above formulas were derived in [3, 4] and quoted in [5]; however they contained sign errors. Here x_{1c} and x_{2c} are the coordinates of the center of gravity of the area ω of the transverse section, and n denotes the number of cutouts in the cross section.

For a singly-connected cross section the formulas for the coordinates of the center of flexure can be obtained from the last two formulas by equating to zero the third of the terms contained within the curly brackets.

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